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Toric Calabi-Yau four-folds dual to Chern-Simons-matter theories

Kazushi Ueda

Department of Mathematics, Graduate School of Science, Osaka University, Machikaneyama 1-1, Toyonaka, Osaka, 560-0043, Japan
E-mail: kazushi@math.sci.osaka-u.ac.jp

Masahito Yamazaki

Department of Physics, University of Tokyo, Hongo 7-3-1, Tokyo, 113-0033, Japan
E-mail: yamazaki@hep-th.phys.s.u-tokyo.ac.jp

Abstract: We propose a new method to find gravity duals to a large class of three-dimensional Chern-Simons-matter theories, using techniques from dimer models. The gravity dual is given by M-theory on $AdS_4 \times Y_7$, where Y_7 is an arbitrary seven-dimensional toric Sasaki-Einstein manifold. The cone of Y_7 is a toric Calabi-Yau 4-fold, which coincides with a branch of the vacuum moduli space of Chern-Simons-matter theories.

Keywords: AdS-CFT Correspondence, M-Theory, Chern-Simons Theories.

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1. Introduction

Recently, there has been considerable developments in understanding duality between four-dimensional $\mathcal{N} = 1$ superconformal quiver gauge theories and type IIB string theory on $AdS_5 \times Y_5$, where Y_5 is a toric Sasaki-Einstein manifold. In particular, using techniques from dimer models (the so-called brane tilings [1 − 3]), we can determine toric Calabi-Yau manifolds (which is a cone of Y_5) dual to quiver gauge theories, and vice versa. This is a fascinating development, since it makes it possible to check AdS/CFT for infinitely many examples, or for arbitrary toric Calabi-Yau 3-folds [3, 4], using for example a-maximization [5] in gauge theory side and volume minimization in gravity side [6].

It is thus natural to consider generalization of all these developements to the case of duality between three-dimensional conformal field theories and M-theory on $AdS_4 \times Y_7$, where Y_7 is now a seven-dimensional toric Sasaki-Einstein manifold. This is not a mere generalization of pure academic interest: First, AdS_4/CFT_3 correspondence should tell us much about the landscape of four-dimensional vacua in string theory with negative cosmological constant, which could be lifted to de Sitter space by various ways (e.g. as in [7]). Second, some of three-dimensional supersymmetric conformal field theories have possible relevance to interesting condensed matter systems. However, this natural generalization has long been a challenging problem, since we did not understand the gauge theory side of this duality; the theory on multiple M2-branes was not known.

This situation changed dramatically quite recently. Bagger, Lambert and Gustavsson [8] proposed a Lagrangian description of the world-volume theory of multiple M2-branes, and their theory is later generalized by [9]. The theory in [9] (the so-called ABJM theory) is a kind of Chern-Simons-matter theory, and is believed to be realized on the world-volume of M2-branes probing $AdS_4 \times S^7/Z_k$, where k is the level of Chern-Simons term.

From our experience in the case of AdS_5/CFT_4, it is natural to generalize the discussion and ask if M-theory on $AdS_4 \times Y_7$ is dual to a certain Chern-Simons-matter theory. In order to preserve supersymmetry, the seven-dimensional manifold Y_7 should be either 3-Sasaki-Einstein, Sasaki-Einstein or weak G_2 manifold, and the corresponding gauge theory (in UV) should have $\mathcal{N} = 3, 2, 1$ supersymmetry, respectively.

In this paper, we consider the case when Y_7 is a toric Sasaki-Einstein manifold.[1] In this case, the corresponding Chern-Simons-matter theories have $\mathcal{N} = 2$, and the vacuum moduli space was analyzed first in [12] and later more generally in [13]. Still, it is not known how to extract toric data of Calabi-Yau 4-fold $C(Y_7)$ (cone of Y_7) given the gauge theory data. In particular, it was not clear how to relate the level of Chern-Simons-matter theories to the coordinates of the lattice points of the toric diagram. In principle, this should be possible since the vacuum moduli space should coincide with $C(Y_7)$.

The primary purpose of this paper is to propose a concrete algorithm to obtain the toric data of the toric Calabi-Yau 4-folds dual to three-dimensional Chern-Simons-matter quiver gauge theories. Our method applies to arbitrary toric Calabi-Yau 4-folds, and utilizes in an essential way some of the techniques from dimer models (brane tilings).

The organization of the present article is as follows. First, in section 2, we briefly review the vacuum moduli space of $\mathcal{N} = 2$ Chern-Simons-matter theories, following [12–14]. In the next section (section 3) we briefly review dimer techniques, and then present our main result, namely the method to extract the toric data of the Calabi-Yau 4-fold from gauge theory data (the bipartite graph and the levels of Chern-Simons). In section 4 we illustrate our method further by studying more examples. The final section is devoted to summary and discussions.

2. Moduli spaces of Chern-Simons-matter theories

Let us begin by recalling some facts about $\mathcal{N} = 2$ superconformal Chern-Simons-matter theories specified from a toric diagram. We will concentrate on the description of vacuum moduli space (VMS), which is discussed first by [12] and later more generally by [13, 14]. Here we mostly follow [13]. In particular, it is expected that (for class of Chern-Simons-matter theories which have gravity duals) the VMS of the Chern-Simons-matter theories dual to M-theory on $AdS_4 \times Y_7$ (where Y_7 is a seven-dimensional Sasaki-Einstein manifold) coincide with the Calabi-Yau 4-fold cone $C(Y_7)$.

We are going to study Chern-Simons-matter theories specified from a quiver diagram. Here a quiver diagram is simply an oriented graph, consisting of vertices (which we denote by $i, j, \ldots \in \mathcal{V}$, where $\mathcal{V}$ denotes the set of vertices) and oriented arrows (which we denote by $a, b, \ldots \in \mathcal{A}$, where $\mathcal{A}$ denotes the set of arrows). Each arrow a has its source and target, which is denoted by $s(a)$ and $t(a)$, respectively. Just as in four-dimensional quiver gauge

[1] See [10] for a discussion in the case of toric hyperKähler manifold. Since $\mathcal{N} = 3$ is a special case of $\mathcal{N} = 2$, our analysis should apply to their case as well. See also [11] for the discussion of squashed S^7, which is an example of weak G_2 manifold with corresponding UV gauge theory having $\mathcal{N} = 1$.

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theories, vertices correspond to gauge groups, and arrows correspond to bifundamentals. Namely, we have a vector multiplet V_i for each $i \in \mathcal{V}$ and a chiral multiplet Φ_a for each $a \in \mathcal{A}$. The gauge group is given by

$$G = \prod_{i=1}^{n} U(N_i), \tag{2.1}$$

where n is the number of vertices of the quiver diagram, and N_i is the rank of the gauge group at vertex i. Since here we are considering Chern-Simons term in three dimensions, we have another parameters k_i (the level of Chern-Simons term) for each vertex i. Also, in this paper we concentrate on the Abelian theories, namely $N_i = 1$ for all i.

Given a quiver diagram, a superpotential and a set of parameters N_i's and k_i's, the corresponding $\mathcal{N} = 2$ Chern-Simons-matter theory is determined. For example, it is possible to write down the explicit form of the Lagrangian. However, since we want to focus on the vacuum moduli space (VMS) of the theory, we here just write down the potential of the theory.

After deleting auxiliary fields, the potential V is given by the sum of F-term potential and D-term potential:

$$V = V_F + V_D, \tag{2.2}$$

where

$$V_F = \sum_{a \in \mathcal{A}} \left| \frac{\partial W}{\partial \phi_a} \right|^2, \tag{2.3}$$

and

$$V_D = \sum_{a \in \mathcal{A}} |\phi_a|^2 (\sigma_{s(a)} - \sigma_{t(a)})^2. \tag{2.4}$$

Here ϕ_a denotes the scalar component of the chiral multiplet Φ_a ($a \in \mathcal{A}$), and σ_i denotes the scalar in the vector multiplet V_i ($i \in \mathcal{V}$). Recall that three-dimensional $\mathcal{N} = 2$ vector multiplet is conveniently obtained by dimensionally reducing $\mathcal{N} = 1$ vector multiplet in four dimensions, and σ_i is the component of the gauge field corresponding to the fourth direction. Also, we have a set of constraints, coming from the equations of motion of auxiliary fields of vector multiplets:

$$\mathcal{D}_i = \frac{k_i \sigma_i}{2\pi}, \tag{2.5}$$

where $\mathcal{D}_i$ is the usual 4d D-term

$$\mathcal{D}_i = -\sum_{a|s(a)=i} |\phi_a|^2 + \sum_{a|t(a)=i} |\phi_a|^2. \tag{2.6}$$

Since nothing is charged under overall diagonal U(1), it follows from (2.6) that

$$\sum_{i=1}^{n} k_i \sigma_i = 0. \tag{2.7}$$

In order to obtain the VMS, we have to minimize the potential, or make the potential vanish. The conditions coming from F-terms are simply

$$\frac{\partial W}{\partial \phi_a} = 0, \tag{2.8}$$

which defines a set

$$\mathcal{Z} = \{dW = 0\} \in \mathbb{C}^{|\mathcal{A}|}. \tag{2.9}$$

This space $\mathcal{Z}$ is sometimes called the "master space" and is recently studied in [15].

We now turn to D-term equations. Interestingly, there are several types of solutions to D-term equations. In this paper, we concentrate on the particular set of solutions defined by

$$\sigma_1 = \sigma_2 = \ldots = \sigma_n \equiv s, \tag{2.10}$$

where $s \in \mathbb{R}$ is arbitrary, although other branches might be interesting. The conjecture by Martelli and Sparks [13] is that this is the branch of the moduli space which becomes a toric Calabi-Yau 4-fold cone $C(Y)$. Our following analysis strongly supports this conjecture.

In this branch, we have

$$\mathcal{D}_i = \frac{k_i s}{2\pi}, \tag{2.11}$$

and (2.7) becomes

$$\sum_{i=1}^{n} k_i = 0. \tag{2.12}$$

This shows that (2.12) is a necessary condition for the branch (2.10) to exist. We will henceforce assume that (2.12) is satisfied.

In the analysis for far, we have completely neglected gauge fields. Now what is special about three dimensions is that a gauge field is dualized (and thus equivalent) to a periodic scalar, and thus should be included when computing the VMS. Of course, basically we can forget about gauge fields since gauge fields are gauged away by gauge transformations. However, this still leaves constant gauge transformations, and after some analysis [12, 13] the result is the following.

Consider a character

$$\chi : \mathrm{U}(1)^n \rightarrow \mathrm{U}(1) \tag{2.13}$$

$$(e^{i\theta_1}, \ldots, e^{i\theta_n}) \mapsto \exp\left(i \sum_{i=1}^{n} k_i \theta_i\right). \tag{2.14}$$

Define a group G_{3d} by

$$G_{3d} = \ker\chi/\mathrm{U}(1), \tag{2.15}$$

where U(1) is the overall diagonal U(1), which is clearly in kerχ by (2.12). Then this group G_{3d} acts as the (effectively acting) group of gauge transformations, and the moduli space of 3d theory is given by

$$\mathcal{M}_{3d} = \mathcal{Z}//G_{3d}. \tag{2.16}$$

This moduli space has an interesting connection to the moduli space of 4d theory [13, 10]. If we divide by the whole $G_{4d} \equiv \mathrm{U}(1)^n/\mathrm{U}(1)$ (here we are again removing overall U(1) which acts trivially on all fields), then we have the moduli space of 4d theory:

$$\mathcal{M}_{4d} = \mathcal{Z}//G_{4d}. \tag{2.17}$$

In order words, we have the relation

$$\mathcal{M}_{4d} = \mathcal{M}_{3d}//\mathrm{U}(1), \tag{2.18}$$

where the U(1) in the last expression is generated by an element of $G_{4d} \equiv \mathrm{U}(1)^n/\mathrm{U}(1)$ which does not belong to $G_{3d} = \ker\chi/\mathrm{U}(1)$. More precisely, $\mathcal{M}_{4d}$ is (for fixed $s \neq 0$) part of the baryonic branch of moduli space of 4d theory (see [16] for recent discussions).

We finish this section with a comment on non-Abelian VMS, although we will concentrate on the Abelian case in all other parts of this paper. If $N_i = N$ for all $i \in \mathcal{V}$, it was shown [13] that VMS becomes the symmetric product of $N = 1$ VMS:

$$\mathcal{M}_{3d,N} = \mathrm{Sym}^N \mathcal{M}_{3d,1}. \tag{2.19}$$

This is consistent with the interpretation that in the dual gravity picture N M2-branes are probing the Calabi-Yau cone $C(Y_7)$.

3. Toric Calabi-Yau 4-folds from Chern-Simons-matter theories

Let us now move to our proposal. As said previously, if the 3d theory is really dual to M-theory on $AdS_4 \times Y_7$, then the VMS of 3d theory $\mathcal{M}_{3d}$ should coincide with Calabi-Yau 4-fold cone $C(Y_7)$. This is important, since it says that the possible candidate for gravity dual Y_7 of 3d theories can be determined in this way.

The problem we want to consider in this paper is to give an efficient procedure to obtain $C(Y_7)$ beginning with a quiver diagram, superpotential and CS levels (k_i's). In this paper, we concentrate on the case where dual Calabi-Yau 4-fold $C(Y_7)$ is toric, which makes the analysis technically tractable but still contains infinitely many highly non-trivial examples.[2]

Now since toric Calabi-Yau 4-folds are specified by a toric polytope, the problem is to extract the toric data beginning with a quiver diagram, a superpotential and a set of levels (k_i's). Of course, as we have stressed, it is in principle possible to solve this problem since $\mathcal{M}_{3d}$ should coincide with $C(Y_7)$. However, it is desirable to have simple combinatorial

[2] The superpotential must satisfy a non-trivial conditions in order for $C(Y_7)$ to be toric. There are some discussions on this in AdS_5/CFT_4 context. For example, it is necessary that each bifundamental should appear exactly twice in the terms of the superpotential, which is sometimes called the toric condition [17].

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procedure to carry this out. In fact, in the context of AdS_5/CFT_4 correspondence, this problem is dubbed the "forward problem" and we have now a well-established combinatorial procedure to do this, using techniques from dimer models, such as Kasteleyn matrices and height functions [1]. We now show in the following that dimer model techniques are also useful for solving the problem at hand.

3.1 Review of $\mathbf{CY}_3$ case

Before describing our proposal, let us briefly review the relevant dimer model techniques. Interested readers are referred to reviews [18, 19] for more details and related topics. For conventions such as orientation of arrows, we basically follow [19].

Quiver gauge theories are usually specified by a quiver diagram, but a bipartite graph on $\mathbb{T}^2$ (the so-called brane tiling) is far more powerful and thus we begin with a bipartite graph. A bipartite graph is a graph consistenting of vertices colored either black or white and vertices connecting different colors. Bipartite graphs we consider will always be written on two-dimensional torus $\mathbb{T}^2$. See the bipartite graph of figure 1 for an example, whose corresponding toric Calabi-Yau 3-fold is often called the Suspended Pinched Point (SPP) in the literature, and its corresponding quiver diagram and toric diagram (of the Calabi-Yau 3-fold $C(Y_5)$) is given in figure 1. The superpotential corresponds to a vertex in the bipartite graph, with $\pm$ sign determined according to the color of the vertex.

On a bipartite graph, we can define a perfect matching as a subset of edges which contains each vertex precisely once. In the case of SPP, we have 6 perfect matchings, as shown in figure 2.

Now choose an arbitrary perfect matching D_0 and fix it as a reference matching. Then if we superimpose D_0 with another matching D, then we have a set of closed lines, which we denote by $D - D_0$. The height function is defined by the (signed) intersection number of $D - D_0$ with α and β cycles of $\mathbb{T}^2$:

$$(h_1(D - D_0), h_2(D - D_0)) \equiv (\langle D - D_0, \alpha\rangle, \langle D - D_0, \beta\rangle). \tag{3.1}$$

Then the intersting observation by [1] is that the convex hull of lattice points $(h_1(D - D_0), h_2(D-D_0))$ now coincides with the toric diagram $\Delta \subset \mathbb{Z}^2$ (there are some ambiguities associated to the choice of D_0 and α, β cycles, but actually Δ is unique up to possible ambiguities of $GL(3, \mathbb{Z})$). This fact is now proven [20] and is often called the "fast-forward problem". The example of this procedure for SPP is shown in figure 2.

In a consistent bipartite graph, it was suggested in [21] that the perfect matching corresponding to a vertex of the toric diagram is unique, and we asume throughout this paper that this is the case. Then for each vertex w_α of the toric diagram Δ, we can find the corresponding perfect matching, which is denoted by D_α. Perfect matchings corresponding to the vertices of Δ are called vertex perfect matchings.

3.2 Our proposal and its proof

We now explain our proposal in detail. Our method gives the toric data of the Calabi-Yau 4-fold from gauge theory data, namely from the bipartite graph (which encodes the quiver diagram and the superpotential) and the levels of Chern-Simons terms.

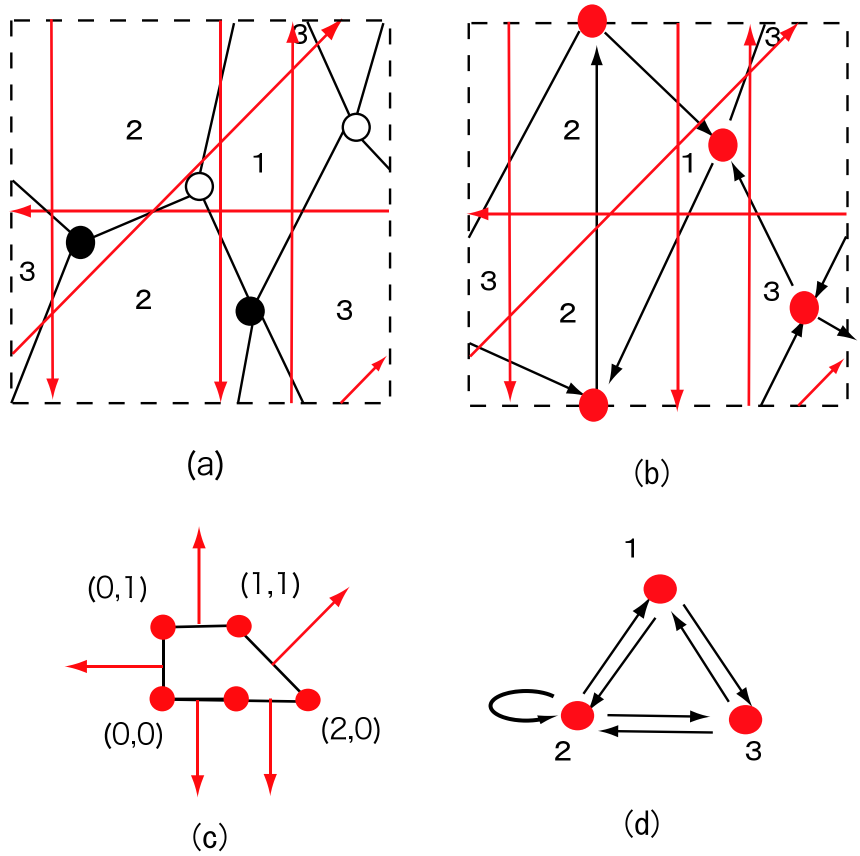

Figure 1: The bipartite graph (a), quiver diagram (b,d) and toric diagram (c) corresponding to the Suspended Pinched Point.

First, we choose four products of paths (bifundamentals) $p_1, \ldots, p_4$ on the bipartite graph as follows. Let p_1 (p_2) be a path on the torus corresponding to α- (β)-cycles of $\mathbb{T}^2$. Let p_4 be a closed path encircling the vertex of the bipartite graph, and thus corresponding to a trivial element of $H^1(\mathbb{T}^2, \mathbb{Z})$. Then the operators $\mathcal{O}_1, \mathcal{O}_2, \mathcal{O}_4$, which correspond to the product of corresponding bifundamentals along paths p_1, p_2, p_4, are invariant under $G_{4d} = \mathrm{U}(1)^n/\mathrm{U}(1)$ and thus gauge invariant. They are usually called mesonic operators. As the remaining path p_3, we take a path (or more preciesly product of paths in general) whose corresponding operator $\mathcal{O}_3$ is invariant under $G_{3d} = \ker\chi/\mathrm{U}(1)$, but not under $G_{4d} = \mathrm{U}(1)^n/\mathrm{U}(1)$.[3] They are charged under baryonic symmetries and thus should be called baryonic operators, although they are not gauge invariant in the usual 4d sense .

Now let us illustrate this method with the example shown in figure 1. We take the levels of Chern-Simons as shown in figure 3. Then χ is given by

$$(e^{i\theta_1}, e^{i\theta_2}, e^{i\theta_3}) \mapsto e^{ik(\theta_1 - \theta_3)}, \tag{3.2}$$

[3]Another way of expressing this fact is that in three dimensions we can attach 't Hooft operators to make them gauge invariant. For example, in the case of ABJM theory [9], p_3 gives the operator C^k discussed in section 2.4 of [9].

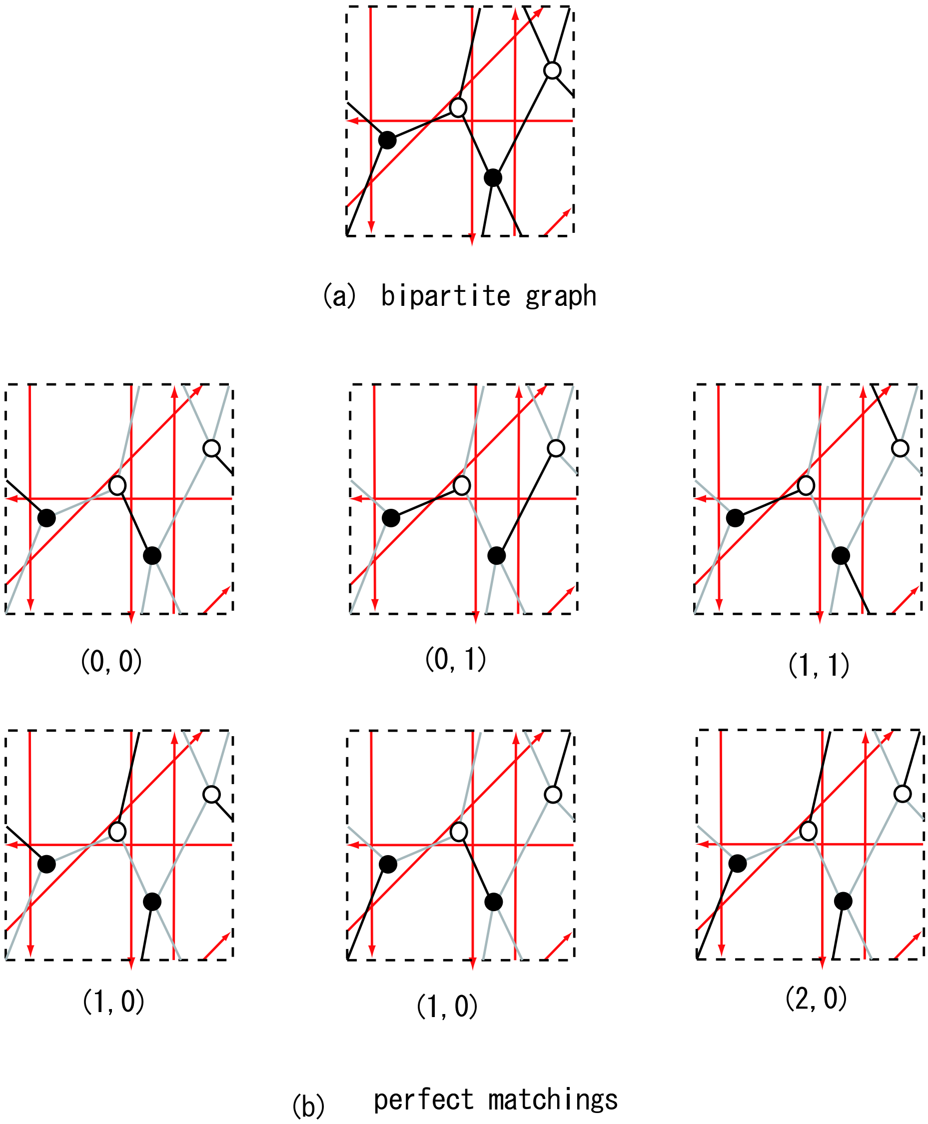

Figure 2: The perfect matchings of bipartite graphs and their height functions. The height functions gives the coordinates of the toric diagram.

and thus kerχ is generated by $(e^{i\theta}, 1, e^{i\theta})$, $(1, e^{i\theta}, 1)$ and discrete subgroup $\mathbb{Z}_k$. In the figure the products of bifundamentals corresponding to p_1, p_2, p_4 are shown as closed paths, and p_3 is the k-th power of the bifundamental shown in the figure. It is easy to check that p_3 is indeed invariant under kerχ. Of course, the choice of path is far from unique, and for example there is in general several choices of α- and β-cycles. We will see, however, that this ambiguity does not matter in the final result.[4]

As you can see from this example, the path p_4 corresponds to a flow of "charges" of levels of Chern-Simons terms (This point is discussed in [22]). Namely, we have a source of

[4]More generally, we take $p_1, \dots p_4$ to be a minimal generating set of operators which are invariant under kerχ.

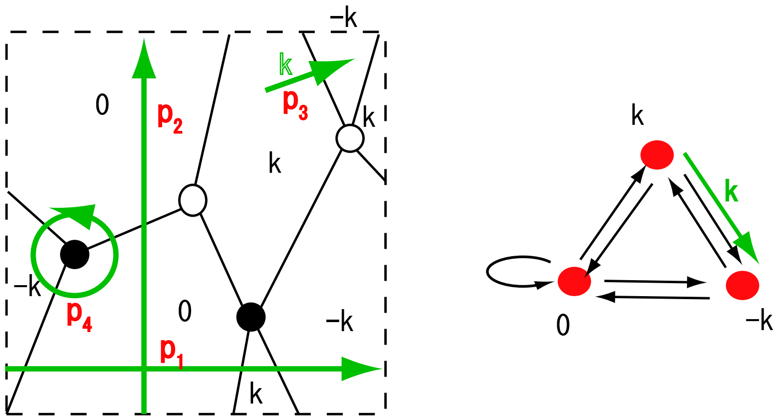

Figure 3: The choices of paths $p_1, \ldots, p_4$ (left) and Chern-Simons levels for SPP (right). On the right figure the flow of the Chern-Simons charge is represented as a green arrow.

k_i units of Chern-Simons levels at the ith face (ith gauge group), and the condition that p_4 is invariant under kerχ is precisely equivalent to the condition that this flow conserves the charge. This often makes the choice of p_4 easier. In the example of SPP, p_4 corresponds to a flow of charge as shown in figure 3.

Now let D_α be the vertex perfect matching corresponding to a lattice point w_α of the toric diagram Δ and let d_α denote the corresponding toric divisor. Let v_i^α $(i = 1, \ldots, 4)$ be the intersection number of p_i with D_α:

$$v_i^\alpha = \langle p_i, D_\alpha \rangle. \tag{3.3}$$

In particular, since p_4 is a path encircling a vertex of the bipartite graph, and since D_α is a perfect matching, v_4^α is always equal to one.

Now our proposal is that the toric polytope of Calabi-Yau 4-fold is given by the convex hull of all lattice points $v^\alpha = (v_1^\alpha, \ldots v_4^\alpha)$. This in particular states that the ambiguity we have encountered in the choice of paths $p_1, \ldots p_4$ does not matter up to $GL(4, \mathbb{Z})$-transformation of the toric polytope. This completes the description of our algorithm to obtain toric data from gauge theory data.

Let us again go back to the example of SPP. In this example, using the paths p_i's shown in figure 3 and perfect matchings shown in figure 2, the spanning vectors of the fan of $C(Y_7)$ are determined to be

$$\begin{aligned} &v_1 = (0,0,0,1), \quad v_2 = (1,0,0,1), \quad v_3 = (2,0,0,1), \\ &v_4 = (0,1,0,1), \quad v_5 = (1,1,k,1). \end{aligned} \tag{3.4}$$

As a Calabi-Yau 4-fold, it corresponds to a certain toric orbifold of $\mathbb{C}^4$, as discussed in [12] (see [23–26] for discussion of orbifolds of BLG models and ABJM models). For more general case of generalized conifolds, completely parallel application of our method again yields the same result as in [12].

Our proposal seems ad hoc at first sight, but in fact we can give a proof of this proposal. To explain that we need two important facts.

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First, in order to determine VMS we have to solve F-term equations. The convenient way to do this is to use perfect matchings. Namely, prepare a field ρ_α for each perfect matching D_α, and then we can solve F-term equations by setting [2]

$$\phi_a = \prod_\alpha \rho_\alpha^{\langle a, D_\alpha \rangle}, \tag{3.5}$$

where ϕ_i is the scalar compoent of chiral superfield Φ_a, and $\langle a, D_\alpha \rangle$ is the intersection number of arrow $a \in \mathcal{A}$ (as written on $\mathbb{T}^2$) and perfect matching D_α. In other words, ρ_α are fields of Gauged Linear Sigma Model (GLSM) [27].

The second important point is that toric divisors d_α (corresponding to $w_\alpha \in \Delta$) is described by an equation $\rho_\alpha = 0$, where ρ_α is the GSLM field introduced in (3.5) (see Theorem 2 and appendix A.2 of [28] for the proof of this fact). Namely, for each toric divisor d, we can associate a subset D of $\mathcal{A}$ of the set of edges of the dimer model so that $a \in D$ if and only if the arrows crossed by a is zero on the divisor d. F-term relations coming from the superpotential implies that D contains a perfect matching, and the condition that d is a divisor (i.e., it has the smallest possible codimension) implies that D is indeed a perfect matching.

Now the gauge invariant operators $\mathcal{O}_1, \ldots, \mathcal{O}_4$ are products of ϕ_a's, and thus are expressed by the product of ρ_α's:

$$\mathcal{O}_i = \prod_\alpha \rho_\alpha^{\langle p_i, D_\alpha \rangle}. \tag{3.6}$$

The gauge invariant operators should span the vacuum moduli space. Mathematically, each operator $\mathcal{O}_i$ of paths defines a $\mathbb{C}^\times$-valued function on $\mathbb{T}^4 \subset \mathcal{M}_{3d}$, which can be extended to a $\mathbb{C}$-valued fuction on the moduli space. By the general theory of toric variety ([29], §3.3, Lemma), the i-th coordinate of the lattice point of the toric diagram corresponding to D_α is exactly the same as the order of zeros of $\mathcal{O}_i$ at divisor d_α. This completes our proof that $\langle p_i, D_\alpha \rangle$ gives the coordinate of the lattice point of the toric diagram.[5]

Note that by forgetting the path p_3, we are naturally lead back to the original story of "fast-forward algorithm" reviewed in section 3.1. Namely, $\langle p_1, D_\alpha \rangle$ $(i = 1, 2)$ coincide with the height functions $h_i(D_\alpha)$ $(i = 1, 2)$ as defined in (3.1). Thus as a direct byproduct our analysis we have given a short alternative proof of the fast-forward algorithm, whose original proof is given in [20].

4. More examples

In the previous section, we have discussed the case where Y_5 is SPP, or more generally generalized conifolds. Although our techniques apply to arbitrary quivers and bipartete graphs corresponding to toric Calabi-Yau 3-folds Y_5, we study $Y^{p,k}(B_4)$ as a good illustrative example.

[5]Since perfect matchings are in one-to-one correspondence with terms in the determinant of Kasteleyn matrix, it automatically follows that we can rephrase our method by using Kasteleyn matrix with suitable weight.

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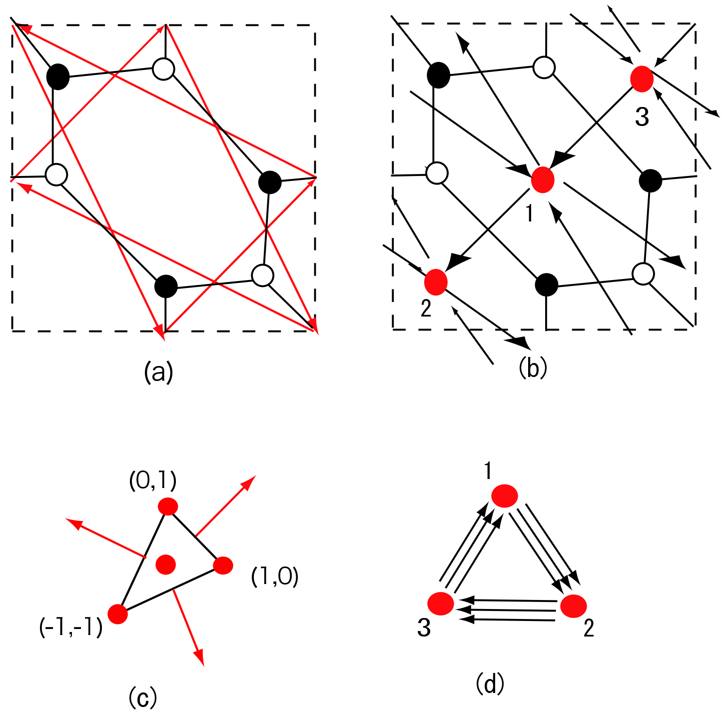

Figure 4: The bipartite graph (a), thee toric diagram (c) and the quiver diagram (b,d) for $C(Y_5) = K_{\mathbb{CP}^2}$.

In [30] it was shown that for any Kähler-Einstein manifold B_4 we can construct explicit metrics on the seven-dimensional Sasaki-Einstein manifold, which are denoted by $Y^{p,k}(B_4)$. Moreover, in version 2 of [13] it was conjectured that the quiver diagram for the Chern-Simons-matter theories exactly coincide with the quiver diagram corresponding to the Calabi-Yau 3-fold K_{B_4}, which is the canonical bundle over B_4. From the classification of Kähler-Einstein manifolds, B_4 is either $\mathbb{CP}^1 \times \mathbb{CP}^1, \mathbb{CP}^2$ or dP_n (n=3,..., 8),[6] where dP_n is the n the del Pezzo surface obtained by blowing up $\mathbb{CP}^2$ at generic points n times. Here we concentrate on the case of $B_4 = \mathbb{CP}^1 \times \mathbb{CP}^1$ and $B_4 = \mathbb{CP}^2$.

4.1 $Y^{p,k}(\mathbb{CP}^2)$

We begin with the case $B_4 = \mathbb{CP}^2$. The dual Chern-Simons-matter quiver is given in figure 4, which is obtained from the toric diagram of $\mathbb{CP}^2$ by standard dimer techniques (see also the recent discussion [14]).

The levels of Chern-Simons terms are denoted by k_1, k_2, k_3, which sum up to zero by (2.12). In this case, it is not difficult to find the paths p_i's, and the final result is that the fan of $C(Y_7)$ is spanned by

$$\begin{aligned} &v_1 = (0,1,0,1), \quad v_2 = (1,0,0,1), v_3 = (-1,-1,k_1-k_3,1), \\ &v_4 = (0,0,k_1,1), \quad v_5 = (0,0,0,1), \quad v_6 = (0,0,-k_3,1). \end{aligned} \tag{4.1}$$

[6]Kähler-Einstein metrics do not exist for dP_1 and dP_2 due to the existence of Matsushima obstruction.

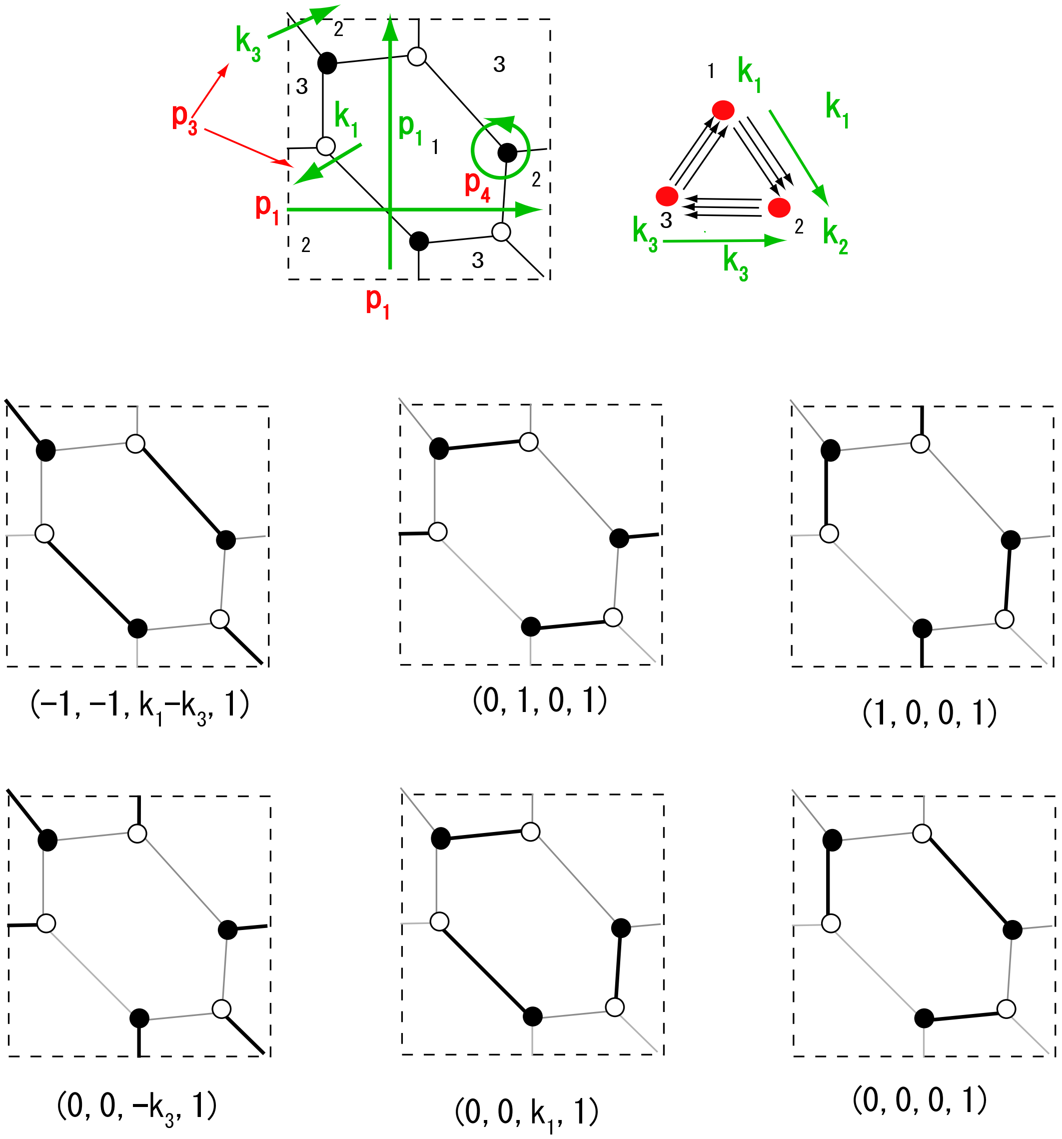

Figure 5: The choice of paths $p_1, \ldots, p_4$, flow of CS cherge (above) and perfect matchings with corresponding coordinate of the lattice point of convex polytope (below) in the case of $C(Y_5) = K_{\mathbb{CP}^2}$.

If you set $k_1 = -p, k_2 = 2p - k, k_3 = -(k_1 + k_2) = k - p$, then the convex polytope $\mathcal{P}$ spanned by $v_1, \ldots, v_5$ exactly coincides with the toric data provided in [31]. Of course, we have another vector v_6, but the condition for v_0 to lie in $\mathcal{P}$ yields the condition $p \leq k \leq 2p$, just as analyzed in version 2 of [13].

4.2 $Y^{p,k}(\mathbb{CP}^1 \times \mathbb{CP}^1)$

Now we treat the case of $B_4 = \mathbb{CP}^1 \times \mathbb{CP}^1$. In this case, there is no previous discussion of Chern-Simons-matter theories in the literature, but the procedure is completely analogous to the case of $B_4 = \mathbb{CP}^2$.

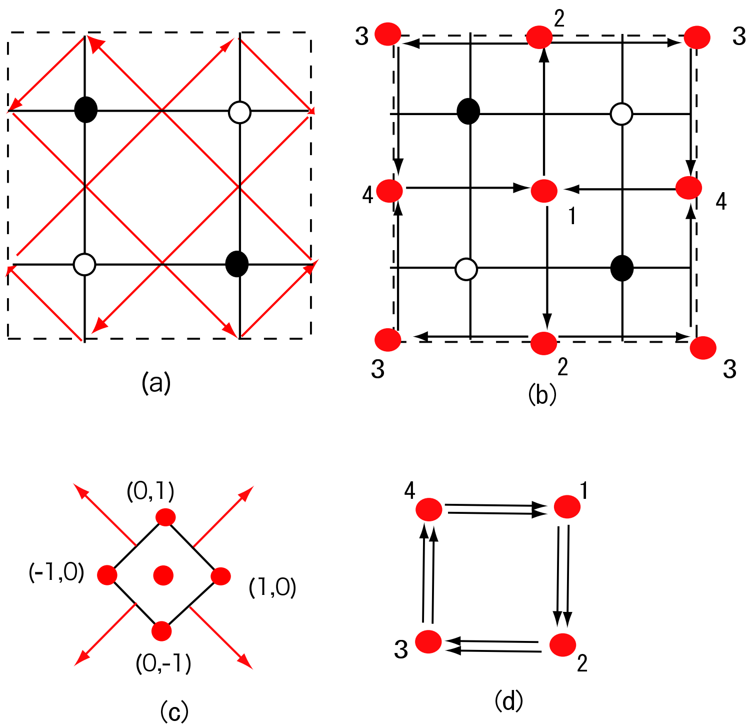

Figure 6: The bipartite graph (a), the toric diagram (c) and quiver diagrams (b,d) for $C(Y_5) = K_{\mathbb{CP}^1\times\mathbb{CP}^1}$.

The bipartite graph and the quiver are shown in figure 6, and the perfect matchings and the choice of paths are shown in figure 7, where l_i's $(i = 1, 2, 3)$ are determined by charge conservation, and thus $l_1 = k_4, l_2 = k_4 + k_1, l_3 = k_4 + k_1 + k_2 = -k_3$.

From figure 7, it is again a straightforward exercise to see that the fan of $C(Y_7)$ is spanned by

$$\begin{aligned} &v_1 = (0,0,0,1), \quad &&v_2 = (0,0,l_1,1), \quad &&v_3 = (-1,0,0,1), \quad &&v_4 = (1,0,l_1+l_3,1), \\ &v_5 = (0,-1,0,1), \quad &&v_6 = (0,1,l_2,1), \quad &&v_7 = (0,0,l_3,1), \quad &&v_8 = (0,0,l_2,1). \end{aligned} \tag{4.2}$$

If we set $l_1 = p, l_2 = k, l_3 = k - p$, namely $k_1 = k - p, k_2 = -p, k_3 = p - k, k_4 = p$, then the spanning vectors are given by

$$\begin{aligned} &v_1 = (0,0,0,1), \quad &&v_2 = (0,0,p,1), \quad &&v_3 = (-1,0,0,1), \quad &&v_4 = (1,0,k,1), \\ &v_5 = (0,-1,0,1), \quad &&v_6 = (0,1,k,1), \quad &&v_7 = (0,0,k-p,1), \quad &&v_8 = (0,0,k,1). \end{aligned} \tag{4.3}$$

The first six vectors span a convex polytope $\mathcal{P}$ which exactly matches with that of [31]. The condition for v_7 to be in $\mathcal{P}$ is given by

$$p \le k \le 2p, \tag{4.4}$$

which again is the same as the condition given by version 2 of [13]. However, we have another vector v_8, which is outside $\mathcal{P}$.

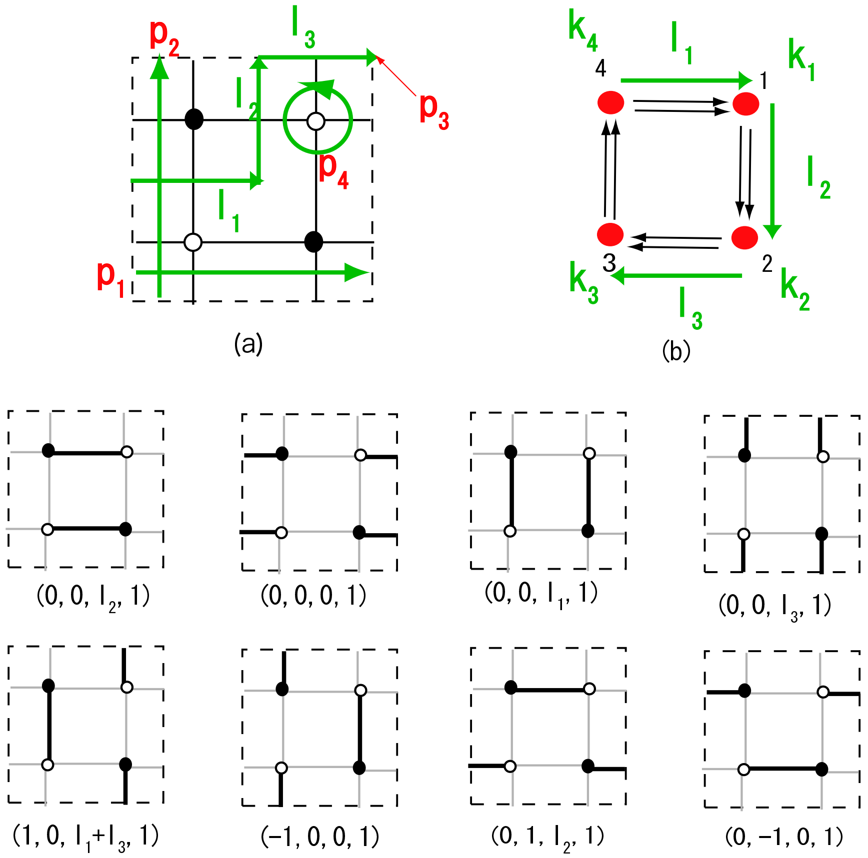

Figure 7: The perfect matchings and choice of paths $p_1, \ldots, p_4$ for $C(Y_5) = K_{\mathbb{CP}^1 \times \mathbb{CP}^1}$.

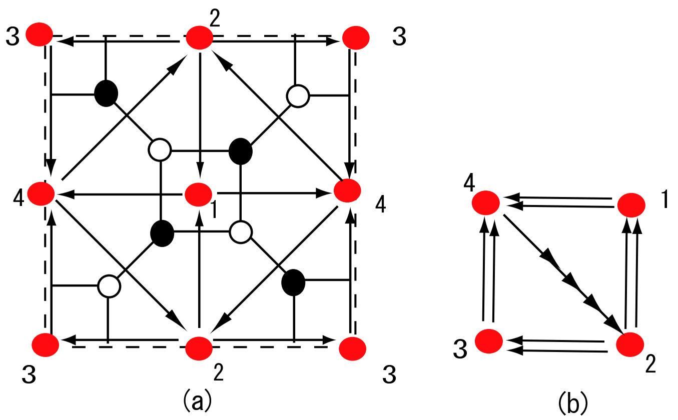

Figure 8: Another bipartite graph (a) and quiver diagram (b) for $C(Y_5) = K_{\mathbb{CP}^1 \times \mathbb{CP}^1}$.

This is still not the end of the story; for $\mathbb{P}^1 \times \mathbb{P}^1$, we have another bipartite graph shown in figure 8. In the context of AdS_5/CFT_4 correspondence, the two bipartite graphs in figures 6 and 8 are related by the Seiberg duality [2]. Since it is not clear whether our algorithm commutes with Seiberg duality, we will also study this another biparite graph explicitly.

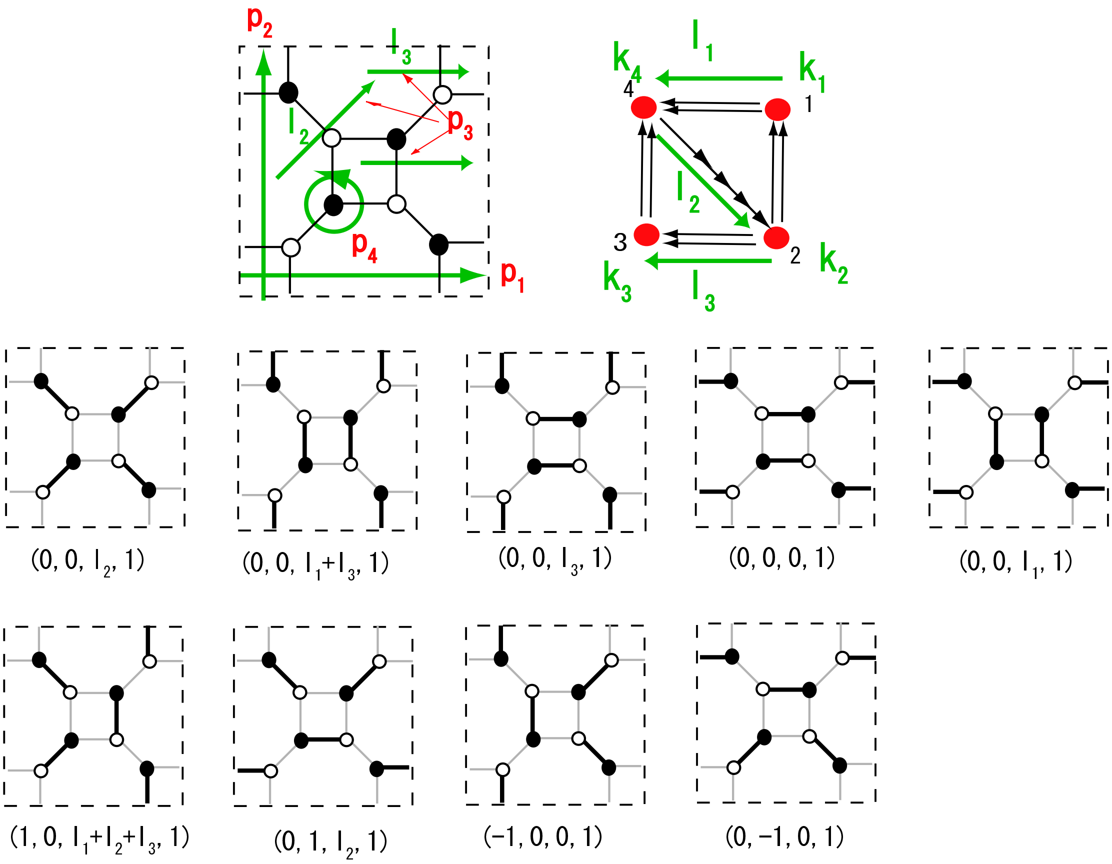

Figure 9: The perfect matchings and choice of paths $p_1, \ldots, p_4$ for $C(Y_5) = K_{\mathbb{CP}^1 \times \mathbb{CP}^1}$.

The procedure to obtain toric data of $C(Y_7)$ is exactly the same as before. After taking appropriate paths $p_1, \ldots, p_4$ and listing all perfect matchings as in figure 9 (from charge conservation we have $l_1 = k_1, l_2 = k_1 + k_4, l_3 = k_1 + k_2 + k_4 = -k_3$), we see that the fan of $C(Y_7)$ is spanned by

$$\begin{aligned}
&v_1 = (0,1,l_2,1), && v_2 = (-1,0,0,1), && v_3 = (0,-1,0,1), && v_4 = (0,0,0,1),\\
&v_5 = (0,0,l_1,1), && v_6 = (0,0,l_3,1), && v_7 = (1,0,l_1+l_2+l_3,1) && v_8 = (0,0,l_2,1),\\
&v_9 = (0,0,l_1+l_3,1).
\end{aligned} \tag{4.5}$$

If we set $l_1 = p, l_2 = k, l_3 = k - p$, namely $k_1 = p, k_2 = -p, k_3 = -k + p, k_4 = k - p$, then the spanning vectors are given by

$$\begin{aligned}
&v_1 = (0,1,k,1), && v_2 = (-1,0,0,1), && v_3 = (0,-1,0,1), && v_4 = (0,0,0,1),\\
&v_5 = (0,0,p,1) && v_6 = (0,0,k-p,1), && v_7 = (1,0,2k,1), && v_8 = (0,0,k,1),\\
&v_9 = (0,0,k,1).
\end{aligned} \tag{4.6}$$

The first five vectors span a subset of convex polytope $\mathcal{P}$ of version 2 of [13], and the condition that v_6 is inside the polytope spanned by $v_1, \ldots, v_5$ again gives $p \leq k \leq 2p$. However, in this case v_7, v_8 and v_9 are outside $\mathcal{P}$. Thus we have shown that the conjecture by [13] does not hold for the case of $B_4 = \mathbb{CP}^1 \times \mathbb{CP}^1$. This also shows that our method of constructing toric data of $C(Y_7)$ does not commute with Seiberg duality in this example. It would be interesting to investigate this point further for more examples.

5. Summary and discussions

In this paper, we provided an explicit procedure to obtain toric data of a Calabi-Yau 4-fold dual to a large class of 3d Chern-Simons-matter theories specified by a bipartite graph (and thus a quiver diagram and a superpotential) and levels of Chern-Simons terms. We used techniques from dimer models, which was also crucial for the short proof of the correctness of our algorithm. We also analyzed the example of $C(Y_5)$ being given by generalized conifolds (SPP), $K_{\mathbb{CP}^2}$, $K_{\mathbb{CP}^1\times\mathbb{CP}^1}$. The analysis in the case of $K_{\mathbb{CP}^1\times\mathbb{CP}^1}$ is inconsistent with the conjecture of [13].

Of course, there are yet many issues to be discussed. For example, straightforward application of our techniques should yield infinitely many, rich set of examples of dual pairs of 3d Chern-Simons-matter theories and Calabi-Yau 4-folds. Using these results, it would be possible to study the issue of Seiberg(-like) duality [32] for Chern-Simons matter theories (see [33] for recent discussions), or the inverse problem of obtaining bipartite graphs from the toric polytope of $C(Y_7)$ (see [21, 34] for AdS_5/CFT_4 case).

At the same time, we should keep in mind that our procedure only gives the possible candidate for the dual gravity theory; even the existence of IR conformal fixed point of 3d theories is not clear in many cases and we are still far from verifying AdS_4/CFT_3 correspondence per se. In the case of AdS_5/CFT_4, we have powerful techniques to analyze 4d $\mathcal{N} = 1$ superconformal field theories, such as NSVZ β-function [35] and a-maximization [5], and it is desireble to find counterparts of these in AdS_4/CFT_3.

We have seen that brane tiling techniques are useful to the AdS_4/CFT_3 correspondence as well. However, the real meaning of dimers in this context is not yet clear, which is in constrast with the fact that brane tilings have a physical meaning as a projection of shapes of D5/NS5 brane configurations [36, 28, 19] in the case of AdS_5/CFT_4. The recent works on the higher-dimensional generalization of dimer models [37] might give a possible clue to this problem.

Of course, we can envisage many generalization and applications, such as generalization to orientifolds [38, 28], fractional branes and cascading [39], non-toric Sasaki-Einstein manifolds [40], marginal deformations [41], application to homological mirror symmetry [42], all of which are analyzed in the AdS_5/CFT_4 context. We hope to return to these topics in the future.

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construction does not commute with Seiberg duality. This work is supported by Grant-in-Aid for Young Scientists (No.18840029) (K. U.), by JSPS fellowships for Youngs Scientists and by World Premier International Research Center InitiativeiWPI Initiative), MEXT, Japan. (M. Y.).

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